Possibly Relevant Translations

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Advances in Modal Logic

Plan for the talk

Develop (possibly relevant) translations.

Sample:

- Corollary results
- Questions for future work
- Conceptual clarifications

Proceed via series of translations:

1. $\{\wedge, \vee, \rightarrow\}$
2. $\{\wedge, \vee, \rightarrow, \circ, \mathsf{t}\}$
3. $\{\wedge, \vee, \rightarrow, \circ, \mathsf{t}, \neg\}$

The basic relevant logic B

Basic relevant logic $\mathbf{B}$ *(or* $\mathbf{B}^+$*):*

Axiom schemes

1. $\varphi \to \varphi$
2. $\varphi_1 \wedge \varphi_2 \to \varphi_i$
3. $\varphi_i \to \varphi_1 \vee \varphi_2$
4. $[(\varphi \to \psi) \wedge (\varphi \to \chi)] \to [\varphi \to \psi \wedge \chi]$
5. $[(\varphi \to \chi) \wedge (\psi \to \chi)] \to [\varphi \vee \psi \to \chi]$
6. $[\varphi \wedge (\psi \vee \chi)] \to [(\varphi \wedge \psi) \vee (\varphi \wedge \chi)]$

Rules

$$\frac{\varphi \qquad \varphi \to \psi}{\psi} \text{ modus ponens}$$

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi} \text{ adjunction}$$

$$\frac{\varphi \to \psi}{(\chi \to \varphi) \to (\chi \to \psi)} \text{ prefixing}$$

$$\frac{\varphi \to \psi}{(\psi \to \chi) \to (\varphi \to \chi)} \text{ suffixing}$$

Axioms for stronger logics

7. $(\varphi \to \psi) \to [(\chi \to \varphi) \to (\chi \to \psi)]$ (prefixing)
8. $(\varphi \to \psi) \to [(\psi \to \chi) \to (\varphi \to \chi)]$ (suffixing)
9. $(\varphi \to \psi) \wedge (\psi \to \chi) \to (\varphi \to \chi)$ (hypothetical syllogism)
10. $\varphi \wedge (\varphi \to \psi) \to \psi$ (pseudo-modus ponens)
11. $[\varphi \to (\varphi \to \psi)] \to (\varphi \to \psi)$ (contraction)
12. $\varphi \to [(\varphi \to \psi) \to \psi]$ (assertion)

Relevant logic $\mathbf{R}$*:*

$$\mathbf{R}^+ = \mathbf{B}^+ \oplus \{7., \dots, 12.\} = \mathbf{B}^+ \oplus \{8., 11., 12.\}$$

Further logics:

$$\mathbf{TW}^+ := \mathbf{B}^+ \oplus \{7., 8.\} \qquad \mathbf{TWJ}^+ := \mathbf{B}^+ \oplus \{7., 8., 9.\}$$

$$\mathbf{C}^+ := \mathbf{B}^+ \oplus \{7., 8., 10.\} \qquad \mathbf{T}^+ := \mathbf{B}^+ \oplus \{7., 8., 11.\}$$

$$\mathbf{RW}^+ := \mathbf{B}^+ \oplus \{7., 8., 12.\}$$

Ordered by strength:

$$\begin{tikzcd}
 & & & \mathbf{RW}^+ \arrow[rrrd, bend left] & & & \\
\mathbf{B}^+ \arrow[r] & \mathbf{TW}^+ \arrow[r] \arrow[rru, bend left] & \mathbf{TWJ}^+ \arrow[r] & \mathbf{C}^+ \arrow[r] & \mathbf{T}^+ \arrow[r] & \mathbf{E}^+ \arrow[r] & \mathbf{R}^+
\end{tikzcd}$$

Routley–Meyer semantics

Routley–Meyer frames: $\mathfrak{F} = (K, R, N)$, where

$$R \subseteq K \times K \times K, \qquad N \subseteq K, \qquad x \leqslant y \text{ iff } \exists n \in N(Rnxy),$$

and:

- (preorder) $\leqslant$ is a preorder,
- (upset) $n \in N$ and $n \leqslant m$ imply $m \in N$,
- (down-down-up) $Rxyz$, $x^- \leqslant x$, $y^- \leqslant y$, $z \leqslant z^+$ imply $Rx^-y^-z^+$.

Routley–Meyer models: $\mathfrak{M} = (\mathfrak{F}, V)$ where

$$x \in V(p) \text{ and } x \leqslant y \text{ imply } y \in V(p).$$

Clauses:

$\mathfrak{M}, x \Vdash p$	iff	$x \in V(p)$,
$\mathfrak{M}, x \Vdash \varphi \wedge \psi$	iff	$\mathfrak{M}, x \Vdash \varphi$ and $\mathfrak{M}, x \Vdash \psi$,
$\mathfrak{M}, x \Vdash \varphi \vee \psi$	iff	$\mathfrak{M}, x \Vdash \varphi$ or $\mathfrak{M}, x \Vdash \psi$,
$\mathfrak{M}, x \Vdash \varphi \to \psi$	iff	$\forall y, z \in K\colon Rxyz$ and $\mathfrak{M}, y \Vdash \varphi$ imply $\mathfrak{M}, z \Vdash \psi$.

Completeness and correspondence

Fact 1: $\mathbf{B}^+$ is complete for all RM-frames.

Notation: Given $R \subseteq K^3$, set $x \cdot y := \{z \mid Rxyz\}$, and lift to sets

$$X \cdot Y := \bigcup_{x \in X, y \in Y} x \cdot y.$$

Fact 2: Each other logic is complete for its correspondence class:

$\mathfrak{F} \Vdash (p \to q) \to [(r \to p) \to (r \to q)]$	iff	$(x \cdot y) \cdot z \subseteq x \cdot (y \cdot z)$	7.
$\mathfrak{F} \Vdash (p \to q) \to [(q \to r) \to (p \to r)]$	iff	$(x \cdot y) \cdot z \subseteq y \cdot (x \cdot z)$	8.
$\mathfrak{F} \Vdash (p \to q) \wedge (q \to r) \to (p \to r)$	iff	$x \cdot y \subseteq x \cdot (x \cdot y)$	9.
$\mathfrak{F} \Vdash p \wedge (p \to q) \to q$	iff	$x \in x \cdot x$	10.
$\mathfrak{F} \Vdash [p \to (p \to q)] \to (p \to q)$	iff	$x \cdot y \subseteq (x \cdot y) \cdot y$	11.
$\mathfrak{F} \Vdash p \to [(p \to q) \to q]$	iff	$x \cdot y \subseteq y \cdot x$	12.

Target modal logic

Language: $\varphi ::= \bot \mid \top \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond(\varphi, \varphi)$.

Fact: $\mathbf{K}_2$ is complete for all frames $\mathbb{F} = (W, R)$, $R \subseteq W^3$.

Diamond clause:

$$\mathbb{M}, w \models \Diamond(\varphi, \psi) \quad \text{iff} \quad \exists v, u \in W\colon\ Rwvu, \mathbb{M}, v \models \varphi, \text{ and } \mathbb{M}, u \models \psi.$$

Observe:

$$\mathbb{M}, w \models \neg\Diamond(\varphi, \neg\psi) \quad \text{iff} \quad \forall v, u \in W\colon\ Rwvu \text{ and } \mathbb{M}, v \Vdash \varphi \text{ imply } \mathbb{M}, u \Vdash \psi.$$

Basic translation

Translate: (abbreviate $\cdot \to \cdot := \neg\Diamond(\cdot, \neg\cdot)$)

$$
\begin{aligned}
p^\star &:= p & (\varphi \vee \psi)^\star &:= \varphi^\star \vee \psi^\star \\
(\varphi \to \psi)^\star &:= \varphi^\star \to \psi^\star & (\varphi \wedge \psi)^\star &:= \varphi^\star \wedge \psi^\star.
\end{aligned}
$$

Theorem

$$\vdash_{\mathbf{B}^+} \varphi \to \psi \qquad \Leftrightarrow \qquad \vdash_{\mathbf{K}_2} \varphi^\star \supset \psi^\star.$$

Proof structure.

Define $F : \{(\mathbb{M}, w)\} \rightleftarrows \{(\mathfrak{M}, x)\} : G$ such that

$$F(\mathbb{M}, w) \Vdash \alpha \qquad \text{iff} \qquad \mathbb{M}, w \vDash \alpha^\star \qquad \text{(hence } \Rightarrow\text{)}$$

and

$$\mathfrak{M}, x \Vdash \alpha \qquad \text{iff} \qquad G(\mathfrak{M}, x) \vDash \alpha^\star \qquad \text{(hence } \Leftarrow\text{)}$$

$\square$

Extending the *-translation

Question 1: Does * extend to translate stronger relevant logics?

Yes: Let $\Phi \subseteq \{7., \ldots, 10.\}$ and $\Phi^\star := \{\alpha^\star \supset \beta^\star \mid \alpha \to \beta \in \Phi\}$. Then:

$$\vdash_{\mathbf{B}^+ \oplus \Phi} \varphi \to \psi \qquad \text{iff} \qquad \vdash_{\mathbf{K}_2 \oplus \Phi^\star} \varphi^\star \supset \psi^\star$$

And no: $\vdash_{\mathbf{B}^+ \oplus \{11.\}} 10.$ but $\nvdash_{\mathbf{K}_2 \oplus \{11.\}^\star} \{10.\}^\star$

Question 2: What if the main connective is not '$\to$'?

Let $\Box_\to \varphi := \top \to \varphi$, and set

$$\begin{aligned} p^* &:= p & (\varphi \vee \psi)^* &:= \varphi^* \vee \psi^* \\ (\varphi \to \psi)^* &:= \Box_\to(\varphi^\star \supset \psi^\star) & (\varphi \wedge \psi)^* &:= \varphi^* \wedge \psi^*. \end{aligned}$$

Then: $\vdash_{\mathbf{B}^+} \varphi \quad \Leftrightarrow \quad \vdash_{\mathbf{K}_2} \varphi^*.$

Proof structure. Now define $F : \{(\mathbb{M}, w)\} \rightleftarrows \{(\mathfrak{M}, n)\} : G$ such that

$$\begin{aligned} F(\mathbb{M}, w) \Vdash \alpha \quad &\text{only if} \quad \mathbb{M}, w \vDash \alpha^\star & \text{(hence } \Rightarrow) \\ \mathfrak{M}, x \Vdash \alpha \quad &\text{if} \quad G(\mathfrak{M}, x) \vDash \alpha^\star & \text{(hence } \Leftarrow) \end{aligned}$$

Lost and found in translation

- *Fails to translate the greater relevant family*
- *(Un)decidability*

Theorem

The modal logic of the class of frames where

$$(x \cdot y) \cdot z \subseteq x \cdot (y \cdot z), \qquad (x \cdot y) \cdot z \subseteq y \cdot (x \cdot z), \qquad x \cdot y \subseteq x \cdot (x \cdot y)$$

is undecidable; and so is the modal logic of frames where

$$(x \cdot y) \cdot z \subseteq x \cdot (y \cdot z), \qquad (x \cdot y) \cdot z \subseteq y \cdot (x \cdot z), \qquad x \in x \cdot x.$$

Proof.

Use $\star$-translation and undecidability of $\mathbf{TWJ}^+ (= \mathbf{B}^+ \oplus \{7., 8., 9.\})$ and $\mathbf{C}^+ (= \mathbf{B}^+ \oplus \{7., 8., 10.\})$, cf. SBK (2025) and Urquhart (1984). $\square$

- *Interpolation*

Recall: $\vdash_{\mathbf{B}^+} \varphi \to \psi \qquad \Leftrightarrow \qquad \vdash_{\mathbf{K}_2} \varphi^\star \supset \psi^\star.$

Translating beyond the basics

Truth t and fusion $\circ$

Extended relevant grammar: $\varphi ::= p \mid \mathsf{t} \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \varphi \circ \varphi$.

Axiomatization of $\mathbf{B}$: Add t as an axiom, and rules

$$\frac{\varphi \to (\psi \to \chi)}{(\varphi \circ \psi) \to \chi} \qquad \frac{(\varphi \circ \psi) \to \chi}{\varphi \to (\psi \to \chi)} \qquad \frac{\varphi}{\mathsf{t} \to \varphi}$$

RM-frames $\mathfrak{F} = (K, R, N)$ and RM-models $\mathfrak{F} = (\mathfrak{M}, V)$ remain the same, with clauses:

$$\begin{array}{lll} \mathfrak{M}, z \Vdash \mathsf{t} & \text{iff} & z \in N, \\ \mathfrak{M}, z \Vdash \varphi \circ \psi & \text{iff} & \exists x, y\colon Rxyz,\ x \Vdash \varphi, \text{ and } y \Vdash \psi. \end{array}$$

Fact: The basic relevant logic remains complete for all frames.

Target modal logic

Extended grammar: $\varphi ::= \bot \mid \top \mid p \mid \mathsf{t} \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond(\varphi, \varphi) \mid \varphi \circ \varphi$,

where t is a *nullary* modality and $\circ$ is a *binary* modality.

$\Rightarrow$ Kripke frames are now $\mathbb{F} = (W, R, S, N)$ where $S \subseteq W^3$ and $N \subseteq W$. But triples (W, R, N) suffice by requiring:

$$(\varphi \circ \psi) \supset \chi \text{ is a theorem} \qquad \text{iff} \qquad \varphi \supset (\psi \to \chi) \text{ is a theorem.}^1$$

The modal clauses are the same as the relevant ones.

$$\begin{aligned} \mathbb{M}, u \vDash \mathsf{t} \quad & \text{iff} \quad u \in N, \\ \mathbb{M}, u \vDash \varphi \circ \psi \quad & \text{iff} \quad \exists w, v\colon Rwvu,\ w \vDash \varphi,\ \text{and } v \vDash \psi. \end{aligned}$$

[4] Or, equivalently, by including:

$$[(p \to q) \circ p] \supset q \qquad \text{and} \qquad p \supset [q \to (p \circ q)].$$

RM-frames are modally definable

Given $\mathbb{F} = (W, R, N)$, recall the def: $x \leqslant y$ iff $\exists n \in N(Rnxy)$.

Proposition (modal correspondences)

Let $\mathbb{F} = (W, R, N)$ be a Kripke frame. Then:

$$
\begin{array}{lcr}
\mathbb{F} \vDash p \supset \mathsf{t} \circ p & \text{iff} & \leqslant \text{ is reflexive,} \\
\mathbb{F} \vDash \mathsf{t} \circ (\mathsf{t} \circ p) \supset \mathsf{t} \circ p & \text{iff} & \leqslant \text{ is transitive,} \\
\mathbb{F} \vDash \mathsf{t} \circ \mathsf{t} \supset \mathsf{t} & \text{iff} & N \text{ is an } \leqslant\text{-upset,} \\
\mathbb{F} \vDash (\mathsf{t} \circ p) \circ q \supset p \circ q & \text{iff} & Rxyz, x^- \leqslant x \Rightarrow Rx^-yz, \\
\mathbb{F} \vDash p \circ (\mathsf{t} \circ q) \supset p \circ q & \text{iff} & Rxyz, y^- \leqslant y \Rightarrow Rxy^-z, \\
\mathbb{F} \vDash \mathsf{t} \circ (p \circ q) \supset p \circ q & \text{iff} & Rxyz, z \leqslant z^+ \Rightarrow Rxyz^+.
\end{array}
$$

In sum, a Kripke frame $\mathbb{F} = (W, R, N)$ is a Routley–Meyer frame iff it validates the listed axioms.

Observe that all formulas are Sahlqvist.

Suggests translating into the least NML validating residuation and the RM-frame axioms (call it $\mathbf{Mod.B}^{+}$)

The $^\circ$-translation

$$
\begin{array}{rcl@{\qquad\qquad}rcl}
p^\circ & := & \mathsf{t} \circ p & (\varphi \vee \psi)^\circ & := & \varphi^\circ \vee \psi^\circ \\
(\varphi \to \psi)^\circ & := & \varphi^\circ \to \psi^\circ & (\varphi \wedge \psi)^\circ & := & \varphi^\circ \wedge \psi^\circ \\
\mathsf{t}^\circ & := & \mathsf{t} & (\varphi \circ \psi)^\circ & := & \varphi^\circ \circ \psi^\circ.
\end{array}
$$

Lemmas:

- For all $\mathbf{Mod.B}^+$-models (W, R, N, V), $\|p^\circ\|$ is a $\leqslant$-upset.
- Let (W, R, N, V) be a $\mathbf{Mod.B}^+$-model. When $V(p)$ is a $\leqslant$-upset:

$$w \vDash p \quad \text{iff} \quad w \vDash p^\circ.$$

Theorem: $\vdash_{\mathbf{B}} \varphi \qquad \text{iff} \qquad \vdash_{\mathbf{Mod.B}^+} \mathsf{t} \supset \varphi^\circ.$

Theorem: For μ^+ the conjunction of res. ax. and the RM-frame ax.:

$$
\begin{aligned}
\mathsf{RMFr}^+(\Gamma) &:= \{\mathfrak{F} \mid \mathfrak{F} \text{ is an RM-frame validating each } \varphi \in \Gamma\} \\
&= \{\mathbb{F} \mid \mathbb{F} \text{ is a } \mathbf{Mod.B}^+\text{-frame validating } \mathsf{t} \supset \varphi^\circ \text{ for each } \varphi \in \Gamma\} \\
&= \{\mathbb{F} \mid \mathbb{F} \text{ is a Kr.-frame validating } \mu^+ \text{ and } \mathsf{t} \supset \varphi^\circ \text{ for each } \varphi \in \Gamma\} \\
&=: \ \mathsf{KFr}^+(\mu^+, \mathsf{t} \supset \Gamma^\circ).
\end{aligned}
$$

Thus, for any class $\mathcal{C}$ of RM-frames: $\quad \mathcal{C} \Vdash \varphi \quad \text{iff} \quad \mathcal{C} \vDash \mathsf{t} \supset \varphi^\circ$

Translating negation $\neg$

- With negation $\neg$, RM-frames become $(K, R, N, *)$, for $* : K \to K$.
- Expand modal grammar with modality $\boxed{\neg}$, so Kripke frames become $(W, R, N, *)$, for $* \subseteq W \times W$.
- Translate $(\neg\varphi)^\circ := \boxed{\neg}\neg\varphi^\circ$.
- Modally define RM-frames:

$\mathbb{F} \models \boxed{\neg}p \subset\supset \neg\boxed{\neg}\neg p$	iff	the relation $*$ is a function,
$\mathbb{F} \models p \subset\supset \boxed{\neg}\boxed{\neg}p$	iff	$*$ is an involution,
$\mathbb{F} \models \boxed{\neg}p \wedge (\mathsf{t} \circ q) \supset \mathsf{t} \circ (q \wedge \boxed{\neg}(\mathsf{t} \circ p))$	iff	$x \leqslant y \;\Rightarrow\; y^* \leqslant x^*$.

- **Theorem:** $\vdash_{\mathbf{B}} \varphi$ iff $\vdash_{\mathbf{Mod.B}} \mathsf{t} \supset \varphi^\circ$.
- **Theorem:** Let μ be μ^+ conjoined with the above. Then:

$$\mathsf{RMFr}(\Gamma) = \mathsf{KFr}(\mu, \mathsf{t} \supset \Gamma^\circ).$$

- **Thus**, for any class $\mathcal{C}$ of RM-frames:

$$\mathcal{C} \Vdash \varphi \qquad \text{iff} \qquad \mathcal{C} \models \mathsf{t} \supset \varphi^\circ$$

Thank you!